Gaussian Filter

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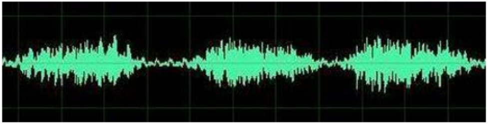
空间域一维信号：



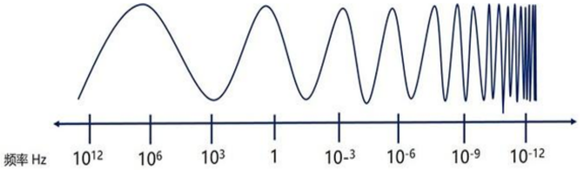
t　$A = f(t)$

空间域二维信号：

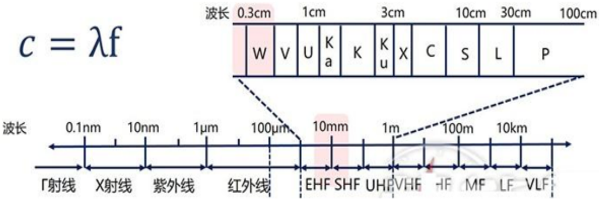
x　$value = f(x, y)$

y

频率域信号：



$$c = \lambda f$$



傅里叶变换：

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt$$

空间域变换到频率域

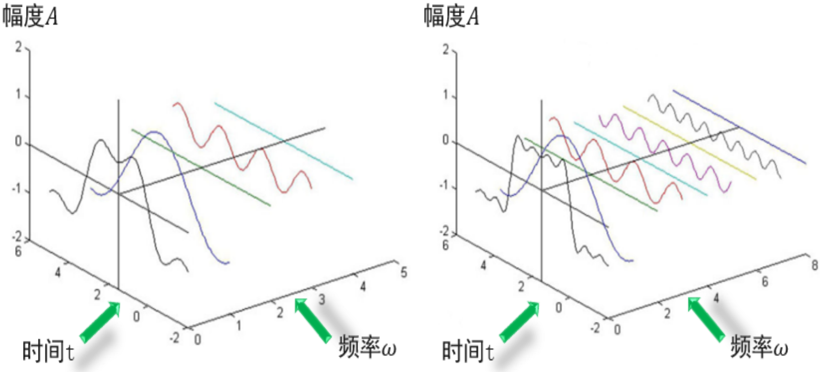
傅里叶逆变换：

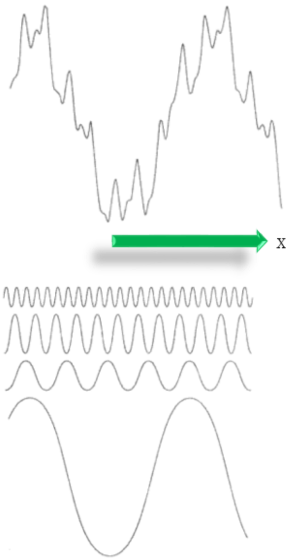
$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{iwt}d\omega$$

频率域变换到空间域

任何周期函数，都可以看作是不同振幅，不同相位正弦波的叠加：

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)\right]$$





x $f(x) = \sin(x) + 0.35\sin(3x) + 0.2\sin(5x) + 0.15\sin(7x)$

$0.15\sin(7x)$

$0.2\sin(5x)$

$0.35\sin(3x)$

$\sin(x)$

如果要去除$0.2\sin(5x)$？

在x的所有定义域$(-\infty,+\infty)$
内都执行如下操作：
$g(x) = f(x) - 0.2\sin(5x)$

- Operator Radius (ρ):
  - gradient change point of the magnitude function

$$f_d(\boldsymbol{w}, \boldsymbol{x}) = \alpha \cdot \frac{\min(\|\boldsymbol{x}\|, \rho)}{\rho} \cdot g(\theta_{(\boldsymbol{w}, \boldsymbol{x})}),$$

  - SphereConv, LinearConv, LogConv have no operator radius

- Boundedness:
  - improves the convergence speed and robustness
  - makes variance of outputs small
  - constrains the Lipschitz constant of neural network, making the entire network more smooth

In particular, a real-valued function f : R → R is called Lipschitz continuous if there exists a positive real constant K such that, for all real x1 and x2,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|.$$

- Smoothness:
  - better approximation rate, more stable, faster convergence
  - more computationally expensive

连续函数卷积： $(f*g)(n) = \int_{-\infty}^{\infty} f(\tau)g(n-\tau)d\tau$

离散函数卷积： $(f*g)(n) = \sum_{\tau=-\infty}^{\infty} f(\tau)g(n-\tau)$

两枚骰子点数相加为4的概率？

f						
	1	2	3	4	5	6

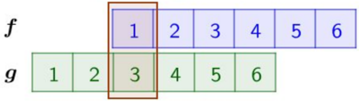
f表示第一枚骰子

$f(1)$表示投出1的概率

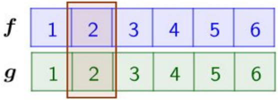
$f(2)$、$f(3)$、$\cdots$以此类推

g						
	1	2	3	4	5	6

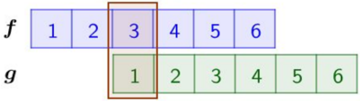
g表示第二枚骰子



出现概率为： $f(1)g(3)$



出现概率为： $f(2)g(2)$



出现概率为： $f(3)g(1)$

$$f(1)g(3) + f(2)g(2) + f(3)g(1)$$

$$(f * g)(4) = \sum_{\tau=1}^{3} f(\tau)g(4-\tau)$$

$$(f * g)(n) = \sum_{\tau=-\infty}^{\infty} f(\tau)g(n-\tau)$$

空间域卷积定理： $F[f_1(t) * f_2(t)] = F_1(\omega) \bullet F_2(\omega)$

空间域卷积的傅里叶变换，等于分别变换到频率域之后的乘积

频率域卷积定理： $F[f_1(t) \bullet f_2(t)] = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

频率域的卷积，等于空间域乘积之后的傅里叶变换

$$F[f_1(t) * f_2(t)] = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau) d\tau \right] e^{-j\omega t} dt$$

傅里叶变换 →

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{+\infty} f_1(\tau) \left[\int_{-\infty}^{+\infty} f_2(t-\tau) e^{-j\omega t} dt \right] d\tau$$

$$= \int_{-\infty}^{+\infty} f_1(\tau) F_2(\omega) e^{-j\omega\tau} d\tau$$

$$= F_2(\omega) \int_{-\infty}^{+\infty} f_1(\tau) e^{-j\omega\tau} d\tau$$

$$= F_2(\omega) F_1(\omega)$$

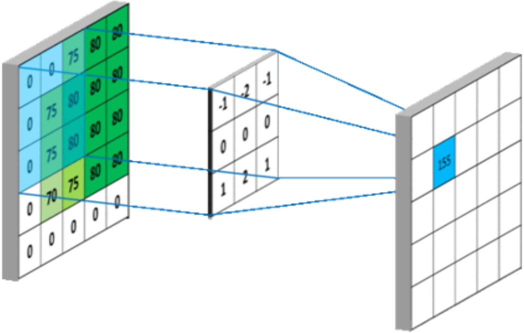
频率域卷积定理： $F[f_1(t) \bullet f_2(t)] = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

a, b的下标相加都为$1, 1$

$$f = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{bmatrix} \quad g = \begin{bmatrix} b_{-1,-1} & b_{-1,0} & b_{-1,1} \\ b_{0,-1} & b_{0,0} & b_{0,1} \\ b_{1,-1} & b_{1,0} & b_{1,1} \end{bmatrix}$$

$$c_{1,1} = a_{0,0}b_{1,1} + a_{0,1}b_{1,0} + a_{0,2}b_{1,-1} + a_{1,0}b_{0,1} + a_{1,1}b_{0,0} + a_{1,2}b_{0,-1} + a_{2,0}b_{-1,1} + a_{2,1}b_{-1,0} + a_{2,2}b_{-1,-1}$$

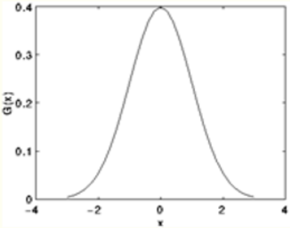
频率域卷积定理：$F[f_1(t)\bullet f_2(t)]=\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$



原图像　　卷积模板　　结果图像

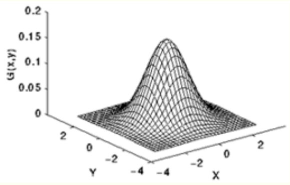
一维高斯函数：

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$



二维高斯函数：

$$G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



正态分布3σ准则：
数值分布在$(\mu - \sigma, u + \sigma)$中的概率为0.6827
数值分布在$(u - 2\sigma, u + 2\sigma)$中的概率为0.9545
数值分布在$(u - 3\sigma, u + 3\sigma)$中的概率为0.9973

$$\frac{1}{273}$$

1	4	7	4	1
4	16	26	16	4
7	26	41	26	7
4	16	26	16	4
1	4	7	4	1

5×5高斯模板

假设高斯模板窗口尺寸为$(2w+1) \times (2w+1)$，
单个像素点运算次数为：
乘法： $(2w+1) \times (2w+1)$
加法： $(2w+1) \times (2w+1) - 1$

$$
\begin{aligned}
G(x,y) &= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \\
&= \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\right) \times \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}\right)
\end{aligned}
$$

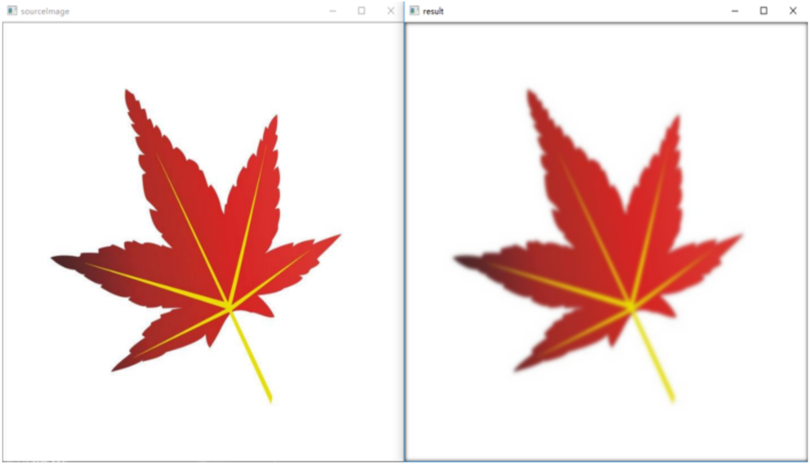
$$
\begin{aligned}
\mathbf{G} &= \frac{1}{S}\begin{bmatrix} g(-w)g(-w) & \dots & g(-w)g(0) & \dots & g(-w)g(w) \\ \vdots & & \vdots & & \vdots \\ g(0)g(-w) & \dots & g(0)g(0) & \dots & g(0)g(w) \\ \vdots & & \vdots & & \vdots \\ g(w)g(-w) & \dots & g(w)g(0) & \dots & g(w)g(w) \end{bmatrix} \\
&= \frac{1}{S}\begin{bmatrix} g(-w) \\ \vdots \\ g(0) \\ \vdots \\ g(w) \end{bmatrix} \times \left[\, g(-w) \dots g(0) \dots g(w) \,\right]
\end{aligned}
$$

先用x方向的一维$(2w+1)$高斯模板卷积，
乘法：　$2w+1$
加法：　$2w$

再用y方向的一维$(2w+1)$高斯模板卷积，
乘法：　$2w+1$
加法：　$2w$

单个像素点的运算次数为：
总共乘法：　$4w+2$ ⟷ 乘法：　$(2w+1)\times(2w+1)$
总共加法：　$4w$ ⟷ 加法：　$(2w+1)\times(2w+1)-1$

高斯滤波能够平滑图像，抑制服从正态分布的噪声（高斯噪声）。



频率域卷积定理： $F[f_1(t) \bullet f_2(t)] = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

a, b的下标相加都为1, 1

$$f = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{bmatrix} \quad g = \begin{bmatrix} b_{-1,-1} & b_{-1,0} & b_{-1,1} \\ b_{0,-1} & b_{0,0} & b_{0,1} \\ b_{1,-1} & b_{1,0} & b_{1,1} \end{bmatrix}$$

$$c_{1,1} = a_{0,0} b_{1,1}$$

频率域卷积定理： $F[f_1(t) \bullet f_2(t)] = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$